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The spectrum of minimal blocking sets

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Abstract

Let $S(q)$ denote the spectrum of minimal blocking sets in a projective plane of order q. Innamorati and Maturo (Ratio Math. 2 (1991) 151–155) proved that if $q \geqslant 4$ then $[2q-1, 3q-5] \cup \{3q-3\} \subseteq S(q)$ and if the plane is Desarguesian then $[2q-1, 3q-3] \subseteq S(q)$. The spectral problem remains to be solved, see Blokhuis (Bull. London Math. Soc. 18 (1986) 132–134); the object of this paper is to study the existence and the uniqueness of certain situations. Several constructions which permit to obtain minimal blocking sets modifying known examples are presented. Moreover, a combinatorial technique to prove the uniqueness of certain configurations realizing largest minimal blocking sets is introduced. The method is applied to the first open case: the uniqueness of a minimal blocking 19-set in PG(2,7). © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction and motivation

A *blocking k-set* in a projective plane π_q of order q is a set K of k points meeting every line but containing no line entirely. A blocking set is said to be *minimal* if it is minimal subject to inclusion, i.e. if for every point $P \in K$ there is a line ℓ such that $\ell \cap K = \{P\}$, such a line is called a *tangent* to K. Moreover, a blocking set of smallest cardinality is said to be *minimum*. Blocking sets, arcs and other remarkable sets are central to the study of the structure of finite projective planes, see [3,11,12]. This is probably because any general projective plane is an unwieldy structure, difficult to handle even by computer, see [10]. The following results are well-known, see [7–9].

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Result 1. *Let K be a minimal blocking k-set in* π_q. *Then*
(a) $q+\sqrt{q}+1 \leqslant k \leqslant q\sqrt{q}+1$;
(b) $k=q+\sqrt{q}+1$ *if and only if q is a square and K is a Baer subplane*;
(c) $k=q\sqrt{q}+1$ *if and only if q is a square and K is a unital.*

The spectrum of a projective plane π_q is the set $S(q)$ of integers k for which there is at least one minimal blocking set of π_q with k points. It is well-known that in the interval of admissible cardinalities of minimal blocking sets, as defined by Result 1(a), there are integers k not belonging to the spectrum. On the spectral problem in [14] the authors proved the following.

Result 2. *In a finite projective plane of order* $q \geqslant 4$ *the spectrum contains the set* $[2q-1, 3q-5] \cup \{3q-3\}$. *Moreover, if the projective plane of order q is Desarguesian then the spectrum contains the interval* $[2q-1, 3q-3]$.

The spectral problem has been studied by many mathematicians, see for instance [5,6,10,11,20,21] but at the moment a complete classification of the configurations which realize minimal blocking sets seems to be impossible. Let us analyze this problem in small planes. In the following, we first summarize the known results showing the spectrum $S(q)$ of minimal blocking sets in π_q.

- PG(2,2) contains no blocking sets, so $S(2)=\emptyset$;
- PG(2,3) contains, up to projective equivalence, exactly one minimal blocking set, the 'standard' example of the three sides of a triangle with the vertices omitted, so $S(3)=\{6\}$;
- PG(2,4) contains exactly three non-isomorphic minimal blocking sets, see [1], the Fano subplane, the classical configuration of the union of two lines, omitting one point on each but including one point on the line joining the omitted points and the Hermitian arc (unital). Hence $S(4)=\{7,8,9\}$.
- For PG(2,5) the complete classification of minimal blocking sets can be found in [2], they are exactly nine and $S(5)=\{9,10,11,12\}$. Of interest is the uniqueness of the case 11.
- PG(2,7) contains, up to isomorphism, exactly two minimum blocking sets, one is the classical projective triangle, see [11], the other one is an interesting geometrical configuration: the dual of an affine plane of order three, see [10,11,15]. The spectrum is the interval $S(7)=[12,19]$. An example of largest minimal case is an interesting configuration called a transitive blocking set, i.e. a Singer 19-cycle on points, see [4,10]; in this paper we prove the uniqueness of this configuration.
- For PG(2,8) the uniqueness of the minimum case is well known. It is given by the classical projective triad, see [10,16].
- It is well known that only two admissible integers 14 and 27 do not belong to the spectrum of the four projective planes of order nine, see [12,13,16].

The object of this paper is to introduce several constructions which permit us to obtain minimal blocking sets by known examples. Moreover, a combinatorial method to prove the uniqueness of large minimal blocking sets is explained through an example. Since the method uses only arithmetical information on points and lines it holds in all finite projective planes. The technique is applied to prove the uniqueness of a minimal blocking 19-set in the projective plane of order seven, PG(2,7).

2. Notation

We assume a basic knowledge of the foundations of finite projective planes. Information and notation may be found in texts such as [11]. From now on let K denote a k-set in a projective plane π_q of order q. Points $P \in K$ are called *inner points* and points $Q \notin K$ are called *outer points*. Let t_i denote the number of lines that intersect K in exactly i points.

We observe that, if K is a minimal blocking set, then $t_0 = t_{q+1} = 0$ and $t_1 \geqslant k$. Moreover, if $k \leqslant 2q-2$ or $k \geqslant 3q-2$ then, in view of results in [15], it follows that $t_{q-1} = t_q = 0$.

The following linear equations on integers $t_i \geqslant 0$, $i = 0, 1, 2, \ldots, q+1$, are well known, see [11]:

$$\sum_{i=0}^{q+1} t_i = q^2 + q + 1,$$

$$\sum_{i=1}^{q+1} it_i = (q+1)k, \tag{2.1}$$

$$\sum_{i=2}^{q+1} i(i-1)t_i = k(k-1).$$

Let P and Q be a point on K and off K, respectively. Let $v_i = v_i(P)$ and $u_i = u_i(Q)$ denote the number of i-secant lines through P and through Q, respectively. The following systems of linear equations are well known, see [11].

$$\sum_{i=1}^{q+1} v_i = q+1,$$

$$\sum_{i=2}^{q+1} (i-1)v_i = k-1, \tag{2.2}$$

$$\sum_{i=0}^{q} u_i = q+1,$$

$$\sum_{i=1}^{q} iu_i = k. \tag{2.3}$$

Obviously, if $t_i=0$ then $v_i(P)=u_i(Q)=0$, for each point $P\in K$ and $Q\notin K$. Moreover, if $\ell=\{P_1,\dots,P_m,Q_1,\dots,Q_{q+1-m}\}$ is a m-secant line of K then

$$\sum_{j=1}^{m} v_i(P_j)+\sum_{h=1}^{q+1-m} u_i(Q_h)=t_i,\quad \forall i\in\{1,2,\dots,q\}-\{m\}, \tag{2.4}$$

$$\sum_{j=1}^{m} v_m(P_j)+\sum_{h=1}^{q+1-m} u_m(Q_h)=t_m+q. \tag{2.5}$$

A k-set K is said to be of *type* $(m_1,\dots,m_r)$ if $t_i\neq 0$ if and only if $i\in\{m_1,\dots,m_r\}$. Moreover, in a k-set of type $(m_1,\dots,m_r)$, an inner point P or an outer point Q is said to be of type $(n_{m_1},\dots,n_{m_r})$ if and only if $v_{m_s}(P)=n_{m_s}$, or $u_{m_s}(Q)=n_{m_s}$, $s=1,2,\dots,r$, respectively.

3. Construction of minimal blocking sets in π_q modifying known examples

In order to study the spectrum of minimal blocking sets in a projective plane in this section we present a method to modify some known examples and to construct minimal blocking sets of new sizes. Let us suppose that K is a minimal blocking k-set. For any $P\in\pi_q$, denote by $\Im_K(P)$ the set of the tangents to K through P and by $\omega(P)$, the weight of P, the number of such tangents, i.e. $\omega(P)=v_1(P)$ if $P\in K$ and $\omega(P)=u_1(P)$ if $P\notin K$. In general, $\forall H\subseteq\pi_q$, let $\Im_K(H)$ and $\omega(H)$ be, respectively, the set of the tangents to K meeting H and their number. Let $I=\{P_1,\dots,P_m\}\subseteq K$ a set of m inner points and $E=\{Q_1,\dots,Q_n\}\subseteq -K$ a set of n outer points.

We have the following

Theorem 1. *If*

(a) $\forall P,\ P'\in I,\ P\neq P',\ PP'\cap(K-I)\neq\emptyset$,

(b) $\forall r\in\Im_K(I),\ r\cap E\neq\emptyset$,

(c) $\forall r\in\pi_q,\ (K-I)\cup E\not\supseteq r$,

then $K'=(K-I)\cup E$ *is a blocking* $(k+n-m)$*-set.*

Proof. Since $I\subseteq K$ and $E\subseteq -K$, we have that $K'=(K-I)\cup E$ is a $(k+n-m)$-set. Let s be a line of π_q. Condition (c) implies that $s\nsubseteq K'$ If $s\cap I=\emptyset$, since K is a blocking set, we have that $s\cap(K-I)\neq\emptyset$ and so $s\cap K'\neq\emptyset$. If $s\cap I\neq\emptyset$, we can have two cases

(I) $s\in\Im_K(I)$,

(II) $s\notin\Im_K(I)$.

If (I) holds, (b) implies that $s\cap E\neq\emptyset$; if (II) holds, by (a) it follows that $s\cap(K-I)\neq\emptyset$. Since $K'=(K-I)\cup E$, we have that $s\cap K'\neq\emptyset$.

Then K' meets every line but contains no line entirely and so it is a blocking set. $\square$

Theorem 2. *Suppose* (a)–(c) *hold. If*

(d) $\forall Q \in E$, $\mathfrak{T}_{K'}(Q) \neq \emptyset$,

(e) $\forall P \in (K - I)$, $\exists r \in \mathfrak{T}_{K-I}(P)$ *such that* $r \cap E = \emptyset$, *then* K' *is a minimal blocking set.*

Proof. The blocking set K' is minimal if and only if through each inner point there passes at least one tangent line to K'. Condition (d) claims that through every point of E there is at least one tangent line to K'; condition (e) implies that through every point of $K - I$ we have at least one tangent line to K'. Since $K' = E \cup (K - I)$ and, by the previous theorem, K' is a blocking set, it follows that K' is a minimal blocking set. □

Example 3. Cases in which $m = 1$. For $n > 1$ the construction allows the size of minimal blocking sets to rise.

Let us suppose that a minimal blocking k-set K contains an inner point P_1 of weight n. Let $r_1, r_2, \ldots, r_n$ be the tangent lines through P_1 and, $\forall i \in \{1, 2, \ldots, n\}$ let Q_i be a point of $r_i - \{P_1\}$. Put $I = \{P_1\}$, $E = \{Q_1, \ldots, Q_n\}$, $K' = (K - I) \cup E$.

(a) and (b) hold. $\forall i \in \{1, 2, \ldots, n\}$, $Q_iP_1 \in \mathfrak{T}_{K'}(Q_i)$ and so (d) holds, too. For $n = 1$ we have (c) iff no line through Q_1 is q-secant K. For $n > 1$ every line is at most $(q + 1 - n)$-secant K and so (c) holds iff $\forall i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, $Q_iQ_j \nsubseteq K'$. In these cases K' is a blocking $(k - 1 + n)$-set.

If every Q_i has weight 1, (e) follows and K' is minimal. This is an improbable condition for small minimal blocking sets, because, in general, the weight of an outer point is greater than the weight of an inner point, but we do not know a similar result. In general (e) follows by the following condition:

$$\forall Q_i\colon \omega(Q_i) \geqslant 2, \quad \forall r \in T_K(Q_i) - \{Q_iP_1\}, \quad \exists \ell \in \mathfrak{T}_{K-I}(r \cap K)\colon \ell \cap E = \emptyset.$$

Example 4. We have a particular case of the previous example if the points of E are collinear in an unique tangent line r through an inner point $P' \in K$ of weight 1. The construction gives a minimal blocking set if and only if line P_1P' is a 2-secant line of K. This leads us to obtain the standard example of large minimal blocking $(3q - 3)$-set 'triangle without vertices' by the standard example of small minimal blocking $2q$-set 'the union of two lines, omitting one point on each but including a point of the line joining the omitted points'.

Example 5. Cases in which $n = 1$. For $m > 1$ the construction allows the size of minimal blocking sets to fall.

Let us suppose that a minimal blocking k-set K contains an outer point Q_1 of weight $\omega \geqslant m$. Let $r_1, r_2, \ldots, r_m$ be the tangent lines through Q_1 and, $\forall i \in \{1, 2, \ldots, m\}$ let $P_i = r_i \cap K$. Put $I = \{P_1, \ldots, P_m\}$, $E = \{Q_1\}$, $K' = (K - I) \cup E$.

(b) holds iff $\forall i \in \{1, 2, \ldots, m\}$, $\omega(P_i) = 1$, (c) holds iff $u_q(Q_1) = 0$. (d) is always true. (e) is true if $m = \omega$, because, in this case, $\forall P \in K - I$, PQ_1 is not a tangent line to K.(a) depend on I.

Example 6. We have a particular case of the previous example if K is the 'triangle without vertices' Q_1RS, with $R,S \notin K$ and $\{P_1,\dots,P_m\} \subseteq RS$. So (b) holds and (a) $\Leftrightarrow m < q-1=\omega(Q_1)$. If such condition holds $\exists P \in (K-I)\cap RS$. Since $u_q(Q_1)=0$, K' is a blocking set. (e) $\Leftrightarrow RS$ is a tangent line of $K' \Leftrightarrow m=q-2$. In this case K' is the 'union of two lines, omitting one point on each but including a point of the line joining the omitted points' and we invert the construction of Example 6.

It is easy to see that the previous constructions give all minimal blocking sets of PG(2,4), starting from a Baer subplane. In PG(2,5), starting from the unique minimum blocking 9-set it is possible to obtain all minimal blocking sets. Since the unique minimal blocking 11-set has any inner point of weight 1, it is not possible, starting from it, by this construction, to obtain the unique minimal blocking 12-set.

The following geometrical construction is of interest.

Theorem 7. *A sufficient condition for the existence of a minimal blocking* $(2q-2)$*-set is that* π_q *contains an affine subplane of order three.*

Proof. Suppose that π_q contains an affine subplane π_3 of order three, with the points A_i, B_i, P_i, $i=1,2,3$ and the lines $A_1A_2A_3, B_1B_2B_3, P_1P_2P_3, A_1B_1P_1$, $A_1B_2P_3$, $A_1B_3P_2$, $A_2B_1P_3$, $A_2B_2P_2$, $A_2B_3P_1$, $A_3B_1P_2$, $A_3B_2P_1$, $A_3B_3P_3$. Let us consider the $(2q-2)$-set of π_q, $H=((a-\{A_1,A_2,A_3\}))\cup(b-\{B_1,B_2,B_3\})\cup\{P_1,P_2,P_3\}$ where a and b are two lines of π_q such that $A_i \in a$, $B_i \in b$, $i=1,2,3$. Set H is a minimal blocking $(2q-2)$-set of π_q. □

In [18] Ostrom and Sherk proved that a Desarguesian projective plane PG$(2,q)$, of order $q=p^h$, contains an affine subplane of order 3 if and only if $q \equiv 1 \bmod 3$ or $p=3$. So an immediate corollary is

Corollary 8. *A sufficient condition for the existence of a minimal blocking* $(2q-2)$*-set in* PG$(2,q)$, *of order* $q=p^h$, *is that* $q\equiv 1 \bmod 3$ *or* $p=3$.

Another geometrical construction is the following.

Theorem 9. *A sufficient condition for the existence of a minimal blocking* $(3q-4)$*-set is that* π_q *contains a proper subplane of order two.*

Proof. Suppose that π_q contains a subplane π_2 of order two, with the points U,V,W, L,M,N,P and the lines $ULW, UMV, VNW, UPN, VPL, WPM$ and LMN. Let $K=UV\cup UW\cup VW-\{U,V,W\}$, where, $\forall A,B \in \pi_q$, AB is the line through A and B. K is a well-known minimal blocking $(3q-3)$-set of π_q, the triangle without the vertices. Since L,M,N are collinear, the $(3q-4)$-set $UV\cup UW\cup VW\cup\{P\}-\{U,V,W,L,M\}$ is a minimal blocking set. □

It is well known that if $q = 9$ we have three non-Desarguesian projective planes containing projective subplanes of order two. Then we have

Corollary 10. *For all order nine planes,* $S(9) \supseteq [17, 24]$.

In [17] Hanna Neumann conjectured that a finite non-Desarguesian projective plane contains projective subplanes of order two. It follows that if Hanna Neumann's conjecture is true then the previous construction gives an example of minimal blocking $(3q-4)$-set in all non-Desarguesian projective planes. So, in view of the previous results, we claim

Theorem 11. *If the Hanna Neumann conjecture is true then in a finite projective plane of order q the spectrum contains the interval* $[2q-1, 3q-3]$.

4. The uniqueness of minimal blocking 19-sets in PG(2,7)

Let K be a minimal blocking set with 19 points in PG(2,7). In this section we will start by calculating the arithmetically admissible cases. In order to do that we consider system (2.1). It gives

$$\begin{aligned} t_1 &= 38 - t_4 - 3t_5 \geqslant 19, \\ t_2 &= -57 + 3t_4 + 8t_5 \geqslant 0, \\ t_3 &= 76 - 3t_4 - 6t_5 \geqslant 0. \end{aligned} \tag{4.1}$$

So $0 \leqslant 3(t_1 - 19) + t_2 = -t_5 \leqslant 0$. This implies that $t_2 = t_5 = 0$ and $t_1 = 19$. Hence, the unique arithmetically admissible case is $(t_1, t_2, t_3, t_4, t_5) = (19, 0, 19, 19, 0)$. We prove

Theorem 12. *K is unique up to isomorphism and it gives rise to a partition of* PG(2,7) *into three isomorphic blocking sets with* 19 *points.*

Proof. In view of system (2.2) we have that any point $P \in K$ is of type $(n_1, n_3, n_4) = (1, 3, 4)$. Moreover, by system (2.3), we have that any point $Q \notin K$ is of type $(n_1, n_3, n_4) \in \{(3, 4, 1), (4, 1, 3)\}$. Let us denote by x the number of points of type (3,4,1) and by y the number of points of type (4,1,3). By counting in two different ways, we have $3x + 4y = 133$ (number of pairs (Q, ℓ) where $Q \notin K$, $Q \in \ell, \ell$ is a tangent line of K) and $4x + y = 95$ (number of pairs (Q, ℓ) where $Q \notin K$, $Q \in \ell, \ell$ is a 3-secant line of K). It follows that the non-negative integers x and y must satisfy the following system of linear equations: $3x + 4y = 133$ and $4x + y = 95$, which gives the unique solution $(x, y) = (19, 19)$. Hence we have two 19-sets of outer points, K_1 of type (3,4,1) and K_2 of type (4,1,3). Since $(t_1, t_2, t_3, t_4, t_5) = (19, 0, 19, 19, 0)$ and K is an irreducible blocking set with 19 points, for any inner point there pass exactly a tangent line, three 3-secant and four 4-secant. Let ℓ be a line. If ℓ is a tangent line the other 18 tangents meet ℓ in outer points. So on ℓ the outer points are exactly three of type (3,4,1) and four of type (4,1,3). If ℓ is a 3-secant line there are other six 3-secant lines meeting ℓ in inner points and 12 3-secant lines meeting ℓ in outer points. So on ℓ the outer points are exactly four of type (3,4,1) and one of type (4,1,3). Finally, if ℓ is a 4-secant line there are other 12 4-secant lines meeting ℓ in inner points and six 4-secant lines meeting ℓ in outer points. So on ℓ the outer points are exactly one of type (3,4,1) and three of type (4,1,3). Then any line ℓ meets each of the disjoint sets K, K_1, K_2 and so they are all minimal blocking sets.

It is well known that PG(2,7) is cyclic, i.e. admits a cyclic group $\sum$ (Singer group), transitive, and hence regular, on the set of all points, see [4,10,19]. So the point-set of PG(2,7) can be identified with Z_{57}, in such a way that $g(x) = x + 1$, where g is a generator of $\sum$. Hence the line-set is $\{\ell + x \bmod 57\colon \ell = 0, 1, 3, 13, 32, 36, 43, 52, x \in Z_{57}\}$. Since 19 is a divisor of 57, we have a Singer subgroup which has exactly three point-orbits, each of length 19 and pairwise equivalent under $\sum$.

One can identify the point set K with the set of all $n \in Z_{57}$ such that $n \equiv 0 \bmod 3$, the point set K_1 with the set of all $n \in Z_{57}$ such that $n \equiv 1 \bmod 3$ and the point set K_2 with the set of all $n \in Z_{57}$ such that $n \equiv 2 \bmod 3$. So the restrictions of g to K, K_1 and K_2 are isomorphisms, respectively, of K onto K_1, of K_1 onto K_2 and of K_2 onto K. $\square$

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